The University of Osaka

Aggregation functions

Vicenç Torra

December 2025

Department of Computing Science, Umeå University, Sweden

Motivation

- Aggregation functions
  - Link between aggregation functions and fuzzy measures
  - Non-additive measures also correspond to monotonic games (in the game theoretic sense / cooperative game theory)
- I will discuss two type of tools
  - Υ-values, with relation to Shapley values as a tool for explainability and to approximate measures
  - Optimal Transport Problem / Wasserstein metric

Outline

1. Aggregation functions
2. Non-additive measures and games
3. Shapley and Υ Values
    - Shapley value
    - Υ values
    - Shapley vs. Υ-values
    - Approximation using distorted probabilities
4. Υ-values and explainability
5. Optimal transport problem
    - The classical problem: for probabilities
    - Our problem: for non-additive measures

Aggregation functions

Some preliminaries

Aggregation functions

- Aggregation functions for numerical data:
  $\mathbb{C}(a_1, \ldots, a_n)$ for $a_i \in [0, 1]$
  - Monotonicity:
  $$\mathbb{C}(a_1, \ldots, a_n) \leq \mathbb{C}(b_1, \ldots, b_n) \text{ if } a_i \leq b_i$$
  - Boundary condition:
  $$\mathbb{C}(a, \ldots, a) = a \text{ (for all or some } a\text{)}$$

Aggregation functions

- Aggregation functions for numerical data:
  $\mathbb{C}(a_1, \ldots, a_n)$ for $a_i \in [0, 1]$
  - Monotonicity:
  $$\mathbb{C}(a_1, \ldots, a_n) \leq \mathbb{C}(b_1, \ldots, b_n) \text{ if } a_i \leq b_i$$
  - Boundary condition:
  $$\mathbb{C}(a, \ldots, a) = a \text{ (for all or some } a\text{)}$$
- Examples:
  - Arithmetic mean: $AM(a_1, \ldots, a_n) = \sum a_i / n$

Aggregation functions

- Aggregation functions for numerical data:
  $\mathbb{C}(a_1, \ldots, a_n)$ for $a_i \in [0, 1]$
  - Monotonicity:
    $$\mathbb{C}(a_1, \ldots, a_n) \leq \mathbb{C}(b_1, \ldots, b_n) \text{ if } a_i \leq b_i$$
  - Boundary condition:
    $$\mathbb{C}(a, \ldots, a) = a \text{ (for all or some } a\text{)}$$
- Examples:
  - Arithmetic mean: $AM(a_1, \ldots, a_n) = \sum a_i/n$
  - Weighted mean: $WM_p(a_1, \ldots, a_n) = \sum p_i a_i$

Aggregation functions

- Aggregation functions for numerical data:
  $\mathbb{C}(a_1, \ldots, a_n)$ for $a_i \in [0, 1]$
  - Monotonicity:
  $$\mathbb{C}(a_1, \ldots, a_n) \leq \mathbb{C}(b_1, \ldots, b_n) \text{ if } a_i \leq b_i$$
  - Boundary condition:
  $$\mathbb{C}(a, \ldots, a) = a \text{ (for all or some } a\text{)}$$

- Examples:
  - Arithmetic mean: $AM(a_1, \ldots, a_n) = \sum a_i/n$
  - Weighted mean: $WM_p(a_1, \ldots, a_n) = \sum p_i a_i$
  - OWA operator: $OWA_w(a_1, \ldots, a_n) = \sum w_i a_{\sigma(i)}$

Aggregation functions

- Aggregation functions for numerical data:
  $\mathbb{C}(a_1, \ldots, a_n)$ for $a_i \in [0, 1]$
  - Monotonicity:
    $$\mathbb{C}(a_1, \ldots, a_n) \leq \mathbb{C}(b_1, \ldots, b_n) \text{ if } a_i \leq b_i$$
  - Boundary condition:
    $$\mathbb{C}(a, \ldots, a) = a \text{ (for all or some } a\text{)}$$

- Examples:
  - Arithmetic mean: $AM(a_1, \ldots, a_n) = \sum a_i/n$
  - Weighted mean: $WM_p(a_1, \ldots, a_n) = \sum p_i a_i$
  - OWA operator: $OWA_w(a_1, \ldots, a_n) = \sum w_i a_{\sigma(i)}$
  - WOWA operator: $WOWA_{w,p}(a_1, \ldots, a_n) = \sum \omega_i a_{\sigma(i)}$, where ω is a function of w and p.

Aggregation functions

- Aggregation functions for numerical data:
  $\mathbb{C}(a_1, \ldots, a_n)$ for $a_i \in [0, 1]$
  - Monotonicity:
    $$\mathbb{C}(a_1, \ldots, a_n) \leq \mathbb{C}(b_1, \ldots, b_n) \text{ if } a_i \leq b_i$$
  - Boundary condition:
    $$\mathbb{C}(a, \ldots, a) = a \text{ (for all or some } a\text{)}$$
- Examples:
  - Arithmetic mean: $AM(a_1, \ldots, a_n) = \sum a_i/n$
  - Weighted mean: $WM_p(a_1, \ldots, a_n) = \sum p_i a_i$
  - OWA operator: $OWA_w(a_1, \ldots, a_n) = \sum w_i a_{\sigma(i)}$
  - WOWA operator: $WOWA_{w,p}(a_1, \ldots, a_n) = \sum \omega_i a_{\sigma(i)}$, where ω is a function of w and p.
  - Choquet integral: ...

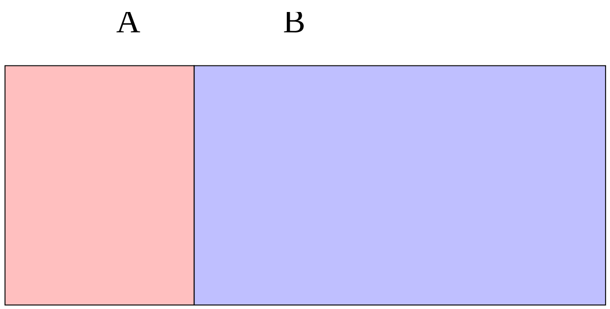
Aggregation functions

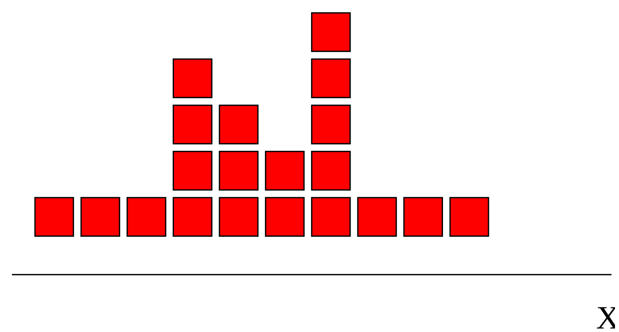
- Considerations on the weights and the aggregation functions
  - Weighted mean:
    - the weights can be seen/are a probability distribution
      The mean is the expected value w.r.t. this probability distribution
      The measure/probability for a set A, $\mu(A) = \sum_{x \in A} p(x)$.
  - OWA, WOWA, Choquet integral:
    - We will come back to this

Non-additive measures and games

(standard) Measures

- Axioms/requirements: set functions: $\mu(A)$ for $A \subseteq X$
  - $\mu(\emptyset) = 0$
  - $\mu(X) = 1$
  - $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \emptyset$
- Examples:
  - Probabilities (with $\mu(X) = 1$)
  - Surface area





Non-additive measures

- Non-additive (fuzzy) measures/monotonic games/capacities are set functions: $\mu(A)$ for $A \subseteq X$
  - $\mu(\emptyset) = 0$
  - $\mu(X) = 1$
  - $\mu(A) \leq \mu(B)$ if $A \subseteq B$

Non-additive measures

- Non-additive (fuzzy) measures/monotonic games/capacities are set functions: $\mu(A)$ for $A \subseteq X$
  - $\mu(\emptyset) = 0$
  - $\mu(X) = 1$
  - $\mu(A) \leq \mu(B)$ if $A \subseteq B$
- We remove the additivity condition (for disjoint A, B) and so

Non-additive measures

- Non-additive (fuzzy) measures/monotonic games/capacities are set functions: $\mu(A)$ for $A \subseteq X$
  - $\mu(\emptyset) = 0$
  - $\mu(X) = 1$
  - $\mu(A) \leq \mu(B)$ if $A \subseteq B$

- We remove the additivity condition (for disjoint A, B) and so
  - We may have (positive/negative) interactions/synergies
    - $\mu(A \cup B) > \mu(A) + \mu(B)$

      or
    - $\mu(A \cup B) < \mu(A) + \mu(B)$

Non-additive measures

- Non-additive (fuzzy) measures/monotonic games/capacities are set functions: $\mu(A)$ for $A \subseteq X$
  - $\mu(\emptyset) = 0$
  - $\mu(X) = 1$
  - $\mu(A) \leq \mu(B)$ if $A \subseteq B$

- We remove the additivity condition (for disjoint A, B) and so
  - We may have (positive/negative) interactions/synergies
    - ▷ $\mu(A \cup B) > \mu(A) + \mu(B)$

      or
    - ▷ $\mu(A \cup B) < \mu(A) + \mu(B)$
  - and it is also possible (for disjoint A, B, C)
    - ▷ $\mu(A) < \mu(B)$ but then $\mu(A \cup C) > \mu(B \cup C)$

Non-additive measures

- Families of measures
  - Additive measures $\mu(A \cup B) = \mu(A) + \mu(B)$ (for $A \cap B = \emptyset$)
  - Symmetric measures
  $$\mu(A) = m(|A|)$$
  - Distorted probabilities
  $$\mu(A) = f(P(A))$$
  - m-dimensional distorted probabilities
  - Hierarchically decomposable fuzzy measures
  - Arbitrary non-additive measures

Non-additive measures

- Families of measures
  - The *number* of distorted probabilities is a small proportion of arbitrary non-additive measures.
  - Number, in terms of possible orders

    (Honda et al. 2002; Narukawa, Torra, 2005)
    - Let $X = \{1, 2, ..., m\}$ we take $\mu(\{1\}) < \mu(\{2\}) \cdots < \mu(\{m\})$.
    - When $|X| = 2$, for DP 2 possible orderings, for arbitrary FM 2.
    - When $|X| = 3$, for DP 2, for arbitrary FM 8. For DP only these orderings are possible:

      $$\mu(\emptyset) < \mu(\{1\}) < \mu(\{2\}) < \mu(\{3\}) < \mu(\{1,2\}) < \mu(\{1,3\}) < \mu(\{2,$$

      $$\mu(\emptyset) < \mu(\{1\}) < \mu(\{2\}) < \mu(\{1,2\}) < \mu(\{3\}) < \mu(\{1,3\}) < \mu(\{2,$$
    - When $|X| = 4$, for DP 14, for arbitrary FM 70016
    - When $|X| = 5$, for DP 546, for arbitrary FM $\mathcal{O}(10^{12})$

Aggregation functions and measures

- Considerations on the weights and the aggregation functions
  - Weighted mean: weights as probability distributions (additive measures)
  - OWA, WOWA, Choquet integral:
    - ▷ OWA: Symmetric measures

$$\mu(A) = m(|A|)$$

    - ▷ WOWA: Distorted probabilities

$$\mu(A) = f(P(A))$$

    - ▷ Choquet integral: Arbitrary non-additive measure

Measures and games

- games are set functions: $\mu(A)$ for $A \subseteq X$
  - $\mu(\emptyset) = 0$

Measures and games

- games are set functions: $\mu(A)$ for $A \subseteq X$
  - $\mu(\emptyset) = 0$

- Example 1: Winning coalitions in a parliament (riksdagen)
  - X parties = `V, S, MP, C, L, KD, M, SD` (.SE 2022)
  - game for any coalition:
    $\mu(coalition) = 1$ if and only if $coalition \geq 175$ seats
  - `mandats = [24, 107, 18, 24, 16, 19, 68, 73]`
    349 seats, 175 are needed to pass a law
    Two examples
    - ▷ $\mu(\{V, S, MP, C, L, KD\}) = 1$
      because $\mathtt{sum}([24, 107, 18, 24, 16, 19]) = 208 \geq 175$
    - ▷ $\mu(\{M\}) = 0$ because $\mathtt{sum}([68]) = 68 < 175$

Measures and games

- games are set functions: $\mu(A)$ for $A \subseteq X$
  - $\mu(\emptyset) = 0$
- Example 1: Winning coalitions in a parliament (riksdagen)
  - X parties = `V, S, MP, C, L, KD, M, SD (.SE 2022)`
  - game for any coalition:
    $\mu(coalition) = 1$ if and only if $coalition \geq 175$ seats
  - `mandats = [24, 107, 18, 24, 16, 19, 68, 73]`
    349 seats, 175 are needed to pass a law
    Two examples
    - $\mu(\{V, S, MP, C, L, KD\}) = 1$
      because $\texttt{sum}([24, 107, 18, 24, 16, 19]) = 208 \geq 175$
    - $\mu(\{M\}) = 0$ because $\texttt{sum}([68]) = 68 < 175$
- Explainability in AI: Game from a model in ML

Shapley and Υ Values

Motivation (to show)

- Given a game/non-additive measure μ
  - The Shapley value provides a summary ϕ
    a summary $O(|X|)$ instead of μ on $O(2^X)$
- Other summaries are possible?
  - Υ-values are a summary

Motivation (to skip)

Motivation

- Motivation 1:
  - Shapley + numerical data is to expectation / weighted mean
  - Υ + numerical data is to linear combination of order statistics?

Motivation (to skip)

Motivation

- Motivation 1:
  - Shapley + numerical data is to expectation / weighted mean
  - Υ + numerical data is to linear combination of order statistics?
- Motivation 2:
  - Shapley are to a probability,
  - Υ are to symmetric measures

Motivation (to skip)

Motivation

- Motivation 1:
  - Shapley + numerical data is to expectation / weighted mean
  - Υ + numerical data is to linear combination of order statistics?
- Motivation 2:
  - Shapley are to a probability,
  - Υ are to symmetric measures
- All this revolves around the concept of non-additive measures (i.e., monotonic games)

Shapley value

Shapley value

- Given X,
  - The set function μ
  - The Shapley value ϕ

- $\phi : X \to \mathbb{R}$
  where $\phi(x)$ is the value associated to $x \in X$
  - The Shapley value is a power index
  - The Shapley value distributes the total gains to the players x,
  - Shapley is an approximation of μ, or summary of μ for x
  - If μ is additive, the approximation is "perfect"
    $$\mu(A) = \sum_{x \in A} \phi(x)$$
  - So, ϕ removes interactions

and the other value

Shapley and Υ-values

- The non-additive (fuzzy) measure perspective
  - Shapley is an approximation of μ,
    power of a variable/element in X

Shapley and Υ-values

- The non-additive (fuzzy) measure perspective
  - Shapley is an approximation of μ,
    power of a variable/element in X
  - Υ is an approximation of μ,
    power of the size of a set in X

Shapley and Υ-values

- The formula perspective. Preliminaries.
  - $A \prec B$ it means that $A \subset B$ and there are no other subsets of X between A and B.
  - Maximal chain on X:
    $$\mathcal{C} = (C_0, C_1, \ldots, C_n) \text{ with } C_i \subseteq X \text{ for } i = 1, \ldots, n.$$
    Then, $\mathcal{C}$ maximal chain of subsets of X if
    $$\emptyset = C_0 \prec C_1 \prec \cdots \prec C_{n-1} \prec C_n = X.$$

Shapley and Υ-values

- The formula perspective. Preliminaries.
  - $A \prec B$ it means that $A \subset B$ and there are no other subsets of X between A and B.
  - Maximal chain on X:
    $$\mathcal{C} = (C_0, C_1, \ldots, C_n) \text{ with } C_i \subseteq X \text{ for } i = 1, \ldots, n.$$
    Then, $\mathcal{C}$ maximal chain of subsets of X if
    $$\emptyset = C_0 \prec C_1 \prec \cdots \prec C_{n-1} \prec C_n = X.$$
- Example: $X = \{x_1, x_2, x_3, x_4\}$
  - $C_0 = \emptyset, C_1 = \{x_4\}, C_2 = \{x_4, x_1\}, C_3 = \{x_4, x_1, x_2\}, C_4 = X$

Shapley and Υ-values

- The formula perspective
  - Shapley is computed as

$$\begin{aligned}\phi_{x_i}(\mu) &= \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n-|S|-1)!}{n!}(\mu(S \cup \{i\}) - \mu(S)) \\ &= \frac{1}{n!} \sum_{\mathcal{C} \in \mathcal{M}(\mathcal{X})} (\mu(\mathcal{C}_{x_i} \cup \{x_i\}) - \mu(\mathcal{C}_{x_i}))\end{aligned}$$

  - Υ-values computed as

$$\begin{aligned}\Upsilon_i &= \frac{1}{|\mathcal{M}(X)|} \sum_{\mathcal{C} \in \mathcal{M}(X)} (\mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1})) \\ &= \frac{1}{n!} \sum_{\mathcal{C} \in \mathcal{M}(X)} (\mu(\mathcal{C}_i) - \mu(\mathcal{C}_{i-1}))\end{aligned}$$

Υ-values and properties

Difference with Shapley

Υ-values: Properties

- Then, the properties:
  - An index $f : G^N \to \mathbb{R}^N$ satisfies the additivity, anonymity, the dummy player condition, and efficiency conditions if an only if f corresponds to the the Shapley value.

Υ-values: Properties

- Then, the properties:
  - An index $f : G^N \rightarrow \mathbb{R}^N$ satisfies the additivity, anonymity, the dummy player condition, and efficiency conditions if an only if f corresponds to the the Shapley value.
- It was possible to characterize the Υ-values also (**Theorem 1**[1]):
  - An index $f : G^N \rightarrow \mathbb{R}^N$ satisfies additivity, absolute anonymity, dummy cardinality, and efficiency if and only if f correspond to the Υ-values.

[1] V. Torra (2024) Υ-values: power indices à la orness for non-additive measures, IEEETFS.

Υ-values: Properties

- What are these properties and how they differ with Shapley?
  - Index f and additivity:
  - when for any pair of measures μ and ν on X it holds that

$$f(\mu + \nu) = f(\mu) + f(\nu).$$

- Property for both Shapley and Υ-values

Υ-values: Properties

- What are these properties and how they differ with Shapley?
  - Index f and efficiency:
  - when for any measure μ on X it holds

$$\sum_{i=1}^{n} f_i = \mu(X)$$

  if $\mu(\emptyset) = 0$; otherwise the total is $\mu(X) - \mu(\emptyset)$.

- Property for both Shapley and Υ-values

Υ-values: Properties

- What are these properties and how they differ with Shapley?
  - Index f, dummy players and dummy cardinality:
  - Dummy player x: $\mu(S \cup \{x\}) = \mu(S) + \mu(\{x\})$ for all $S \subseteq (X \setminus \{x\})$.
  - f dummy player property: for all dummy players $x \in X$, $f_{i(x)}(\mu) = \mu(\{x\})$ for all measures μ.

Υ-values: Properties

- What are these properties and how they differ with Shapley?
  - Index f, dummy players and dummy cardinality:
  - Dummy player x: $\mu(S \cup \{x\}) = \mu(S) + \mu(\{x\})$ for all $S \subseteq (X \setminus \{x\})$.
  - f dummy player property: for all dummy players $x \in X$, $f_{i(x)}(\mu) = \mu(\{x\})$ for all measures μ.
  - Dummy cardinality i: $\mu(S \cup \{x\}) = \mu(S) + h_i$ for all $|S \cup \{x\}| = i$, s.t. $S \subseteq (X \setminus \{x\})$, and h_i a constant.
  - f dummy cardinality property: for all dummy cardinality i, $f_i(\mu) = h_i$ for all measures μ.

Υ-values: Properties

- What are these properties and how they differ with Shapley?
  - Index f, dummy players and dummy cardinality:
  - Dummy player x: $\mu(S \cup \{x\}) = \mu(S) + \mu(\{x\})$ for all $S \subseteq (X \setminus \{x\})$.
  - f dummy player property: for all dummy players $x \in X$, $f_{i(x)}(\mu) = \mu(\{x\})$ for all measures μ.
  - Dummy cardinality i: $\mu(S \cup \{x\}) = \mu(S) + h_i$ for all $|S \cup \{x\}| = i$, s.t. $S \subseteq (X \setminus \{x\})$, and h_i a constant.
  - f dummy cardinality property: for all dummy cardinality i, $f_i(\mu) = h_i$ for all measures μ.
- Shapley: dummy player property; Υ: dummy cardinality

Υ-values: Properties

- What are these properties and how they differ with Shapley?
  - Index f, anonymity and absolute anonymity
  - f satisfies the anonymity property when given a permutation π on X, we have
    $$f(\mu^{\pi}) = \pi^{*}(f(\mu))$$
    where $\pi^{*}(x) = (\pi^{*}_{\pi(1)}(x), \ldots, \pi^{*}_{\pi(|X|)}(x))$ and $\pi^{*}_{\pi(k)}(x) = x_K$.

Υ-values: Properties

- What are these properties and how they differ with Shapley?
  - Index f, anonymity and absolute anonymity
  - f satisfies the anonymity property when given a permutation π on X, we have
$$f(\mu^\pi) = \pi^*(f(\mu))$$
where $\pi^*(x) = (\pi^*_{\pi(1)}(x), \ldots, \pi^*_{\pi(|X|)}(x))$ and $\pi^*_{\pi(k)}(x) = x_K$.
  - f satisfies the absolute anonymity property when given a permutation π on X, we have
$$f(\mu^\pi) = f(\mu).$$

- Shapley: anonymity property; Υ: absolute anonymity property

Approximation using distorted probabilities

Summaries of μ, and approximations of μ

- Additive
  - If μ additive (e.g. probability), Shapley value is the probability
- Symmetric measures
  - if $\mu(A) = \mu(B)$ when $|A| = |B|$
    (only set size matters)
  - Then, (characterization) there exist $w_1, \ldots, w_n$ such that
    $$\mu(A) = \sum_{i=1}^{|A|} w_i$$
  - If μ is symmetric, $\Upsilon_i = w_i$
- If μ is a distorted probability,
  approximation by $\Upsilon_i = w_i, P = \phi$

Summaries of μ, and approximations of μ

- How the approximation is built? $\mu' = f \circ P$
  - Compute $< \phi, \Upsilon >$ from the measures
  - $P := \phi$ (i.e., ϕ represents the probabilities)
  - $w^*(\alpha) = f(k) = \sum_{i=1}^{k} w_i = \sum_{i=1}^{k} \Upsilon_i$
    for $\alpha = |A|/|X| = k/n$
  - Complete w^* interpolating[2] the points already established.
  - $\mu' = w^* \circ \phi = f \circ P$.

- NOTE:
  This is WOWA's interpolation approach from weights p and w.

[2]Interpolation with a continuous monotonic function

Approximations of μ as a way to visualize the measure

- Arbitrary non-additive measures are not easy to grasp
  - Measures are defined by $2^{|X|}$ values
  - Distorted probabilities are interpretable in terms of p and f

- Interpretation of an arbitrary measure through a distorted probability

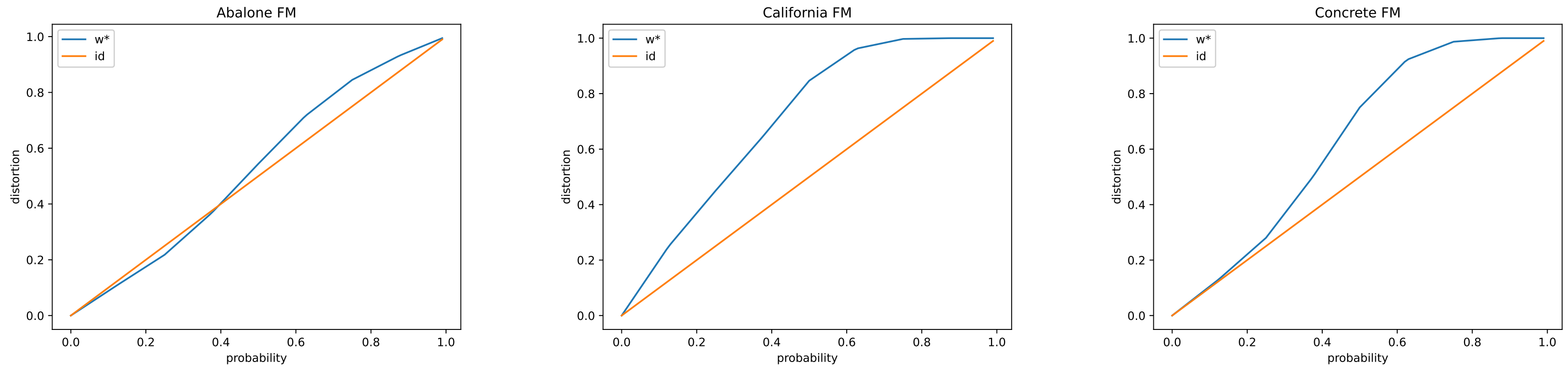
Approximations of μ as a way to visualize the measure

- Some examples.
  - Learn an arbitrary fuzzy measure
    (using a Choquet integral-based ML model)
  - Approximate the measure by a distorted probability

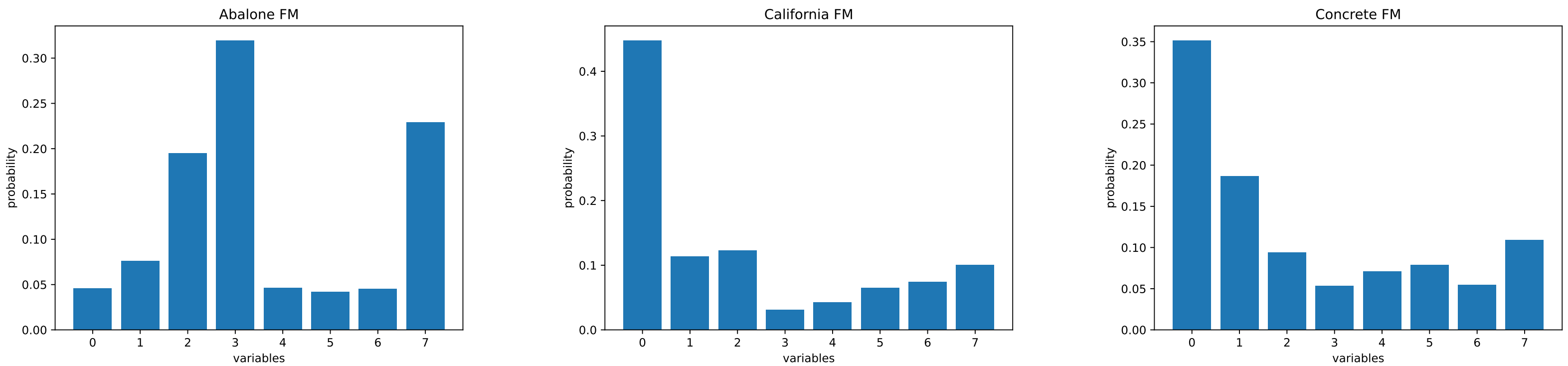
- Data used:
  - Abalone. 4177 records, 8 inputs, 1 output.
  - California. 20640 records, 8 inputs, 1 output.
  - Concrete. 1030 records, 8 inputs, 1 output.

Approximations of μ as a way to visualize the measure

Distortion functions w^* associated to Υ values.



Probability distributions associated to Shapley values.

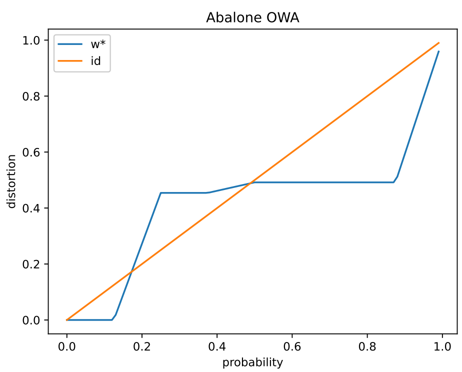


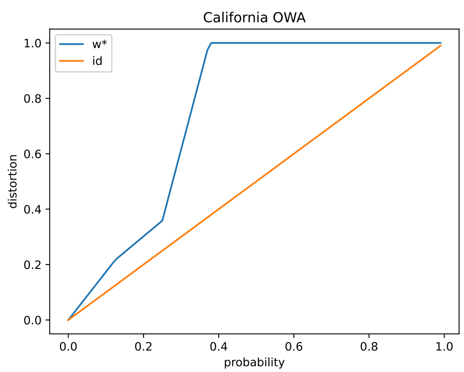
Approximations of μ as a way to visualize the measure

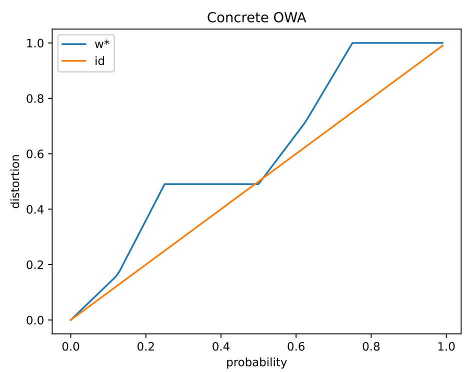
- Properties of the approximation (from ϕ and Υ)

Data set	orness Υ	orness w
Abalone	0.533	0.411
California	0.735	0.796
Concrete	0.653	0.621

Distortion functions w^* associated to optimal OWA parameters w.







Additive and symmetric measures

- If we compute $CI_\mu(f)$, approximations
  - through Shapley, (expectation) and the weighted mean

$$wm_\phi(f)$$

  - through Υ, linear combination of order statistics (owa)

$$owa_\Upsilon(f)$$

Υ-values and explainability

Υ-values and explainability

- Model M at instance u
  - Game: $\mu(S) = M(u^S) - M(u^\emptyset)$
  - Shapley: compute ϕ_x rellevance and importance for $x \in X$
  - $M(u) = \mu(X) + M(u^\emptyset) = \sum_x \phi_i(\mu) + M(u^\emptyset)$
  - Υ: compute Υ_i rellevance and importance[3] for i
    - $M(u) = \mu(X) + M(u^\emptyset) = \sum_i \Upsilon_i(\mu) + M(u^\emptyset)$
    - Also, a summary of the information in M at u.

[3] V.Torra, The complementary role of Shapley and Υ-values in explainability, submitted.

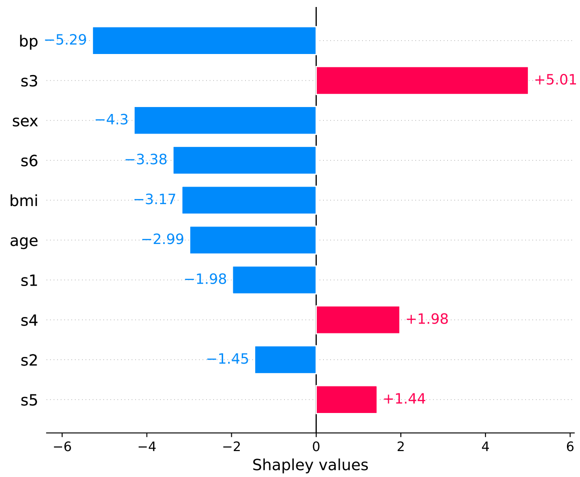
Υ-values and explainability

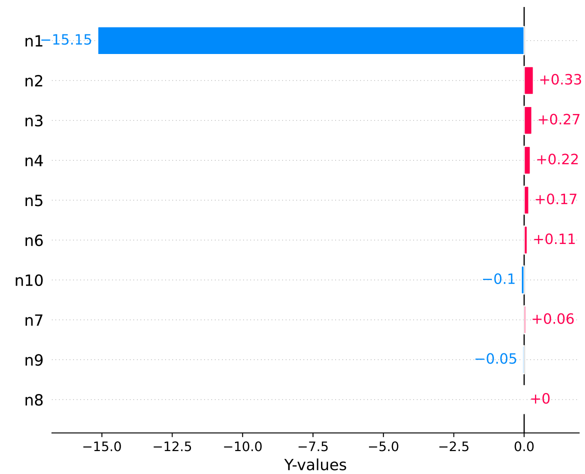
- Model M at instance u
  - Game: $\mu(S) = M(u^S) - M(u^\emptyset)$
  - Shapley: compute ϕ_x rellevance and importance for $x \in X$
  - $M(u) = \mu(X) + M(u^\emptyset) = \sum_x \phi_i(\mu) + M(u^\emptyset)$
  - Υ: compute Υ_i rellevance and importance[3] for i
    - ▷ $M(u) = \mu(X) + M(u^\emptyset) = \sum_i \Upsilon_i(\mu) + M(u^\emptyset)$
    - ▷ Also, a summary of the information in M at u.
  - Summary in Shapley $\neq$ summary in Υ of the same model

[3] V.Torra, The complementary role of Shapley and Υ-values in explainability, submitted.

Υ-values and explainability

- Examples
  - Summary in Shapley ≠ summary in Υ of the same model

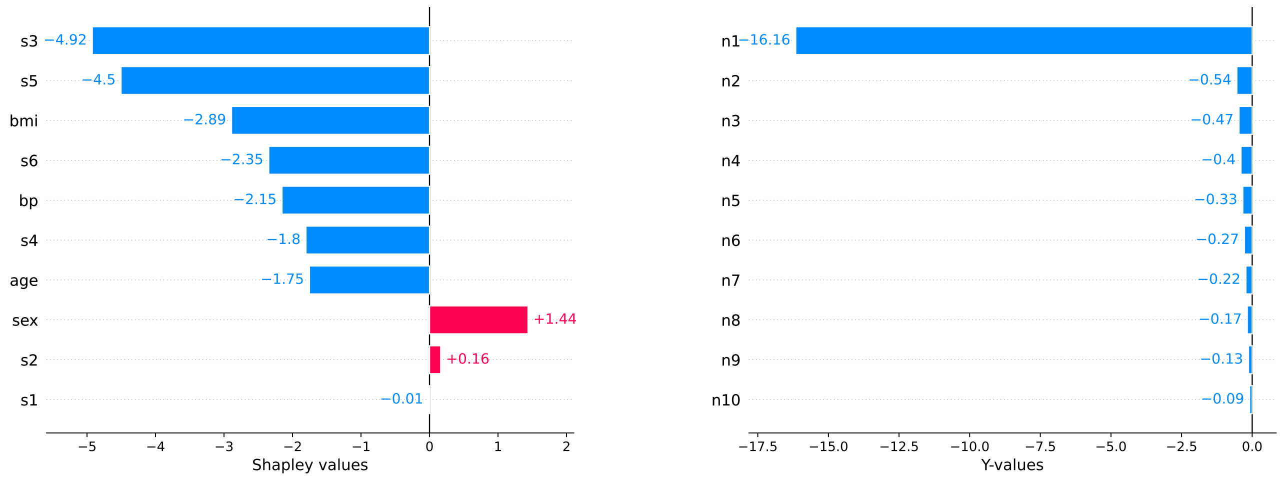




Shapley (left) and Υ-values (right) for a record of the Diabetes data set (record 1 in the test set) computed from a model extracted using SVM/SVR.

Υ-values and explainability

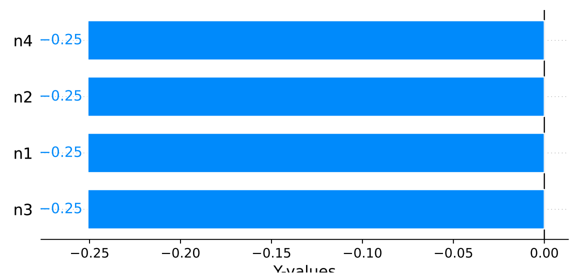
- Examples
  - Summary in Shapley $\neq$ summary in Υ of the same model



Shapley (left) and Υ-values (right) for a record of the Diabetes data set (record 4 in the test set) computed from a model extracted using SVM/SVR.

Υ-values and explainability

- Properties
  - Summary in Shapley $\neq$ summary in Υ of the same model
  - Υ can show the (non-)linearity of the model, Shapley cannot
  - If the model is linear,
    - Shapley = coefficients
    - $\Upsilon = 1/|X|$
  - Example:



Explanations via Υ-values for a record in the Iris data set with output in class 1, and a linear regression model.

Optimal transport problem

Classical problem

- Optimal transport problem between two probabilities
- Wasserstein distance is based on the optimal transport

The classical problem: Optimal transport for probabilities

Classical problem

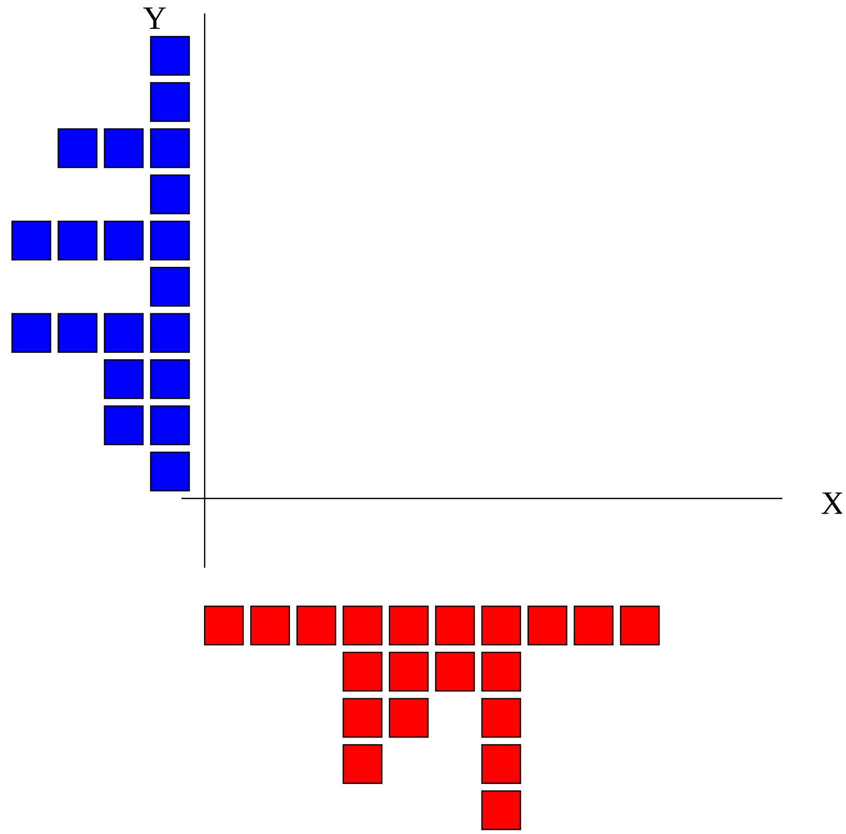
Optimal transport problem: The case of probabilities

- Inputs:
  - X, and probability measure P on X (with prob. dist. p)
  - Y, and probability measure Q on Y (with prob. dist. q)
    (on X and subsets of X, assume X finite)
- Output:
  - Assignment from P to Q
  - A cost of the assignment: optimal

Classical problem

Optimal transport problem: The case of probabilities

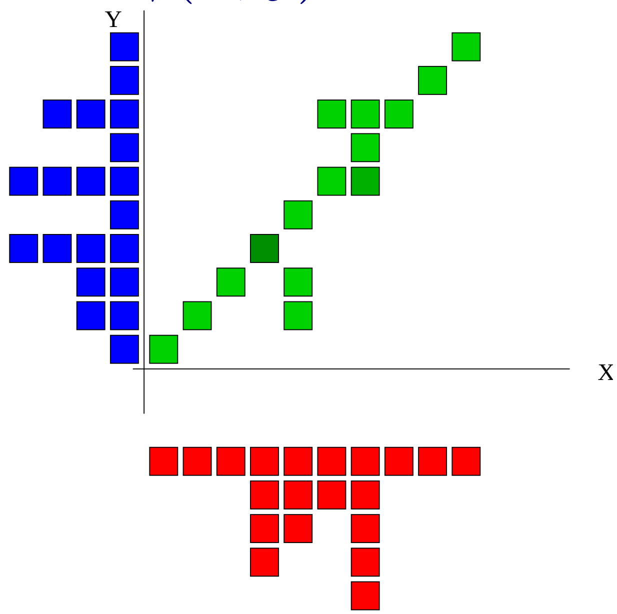
- Probability distributions on X and Y



Classical case

Optimal transport problem: The case of probabilities

- Assignment of probabilities $\gamma(x, y)$



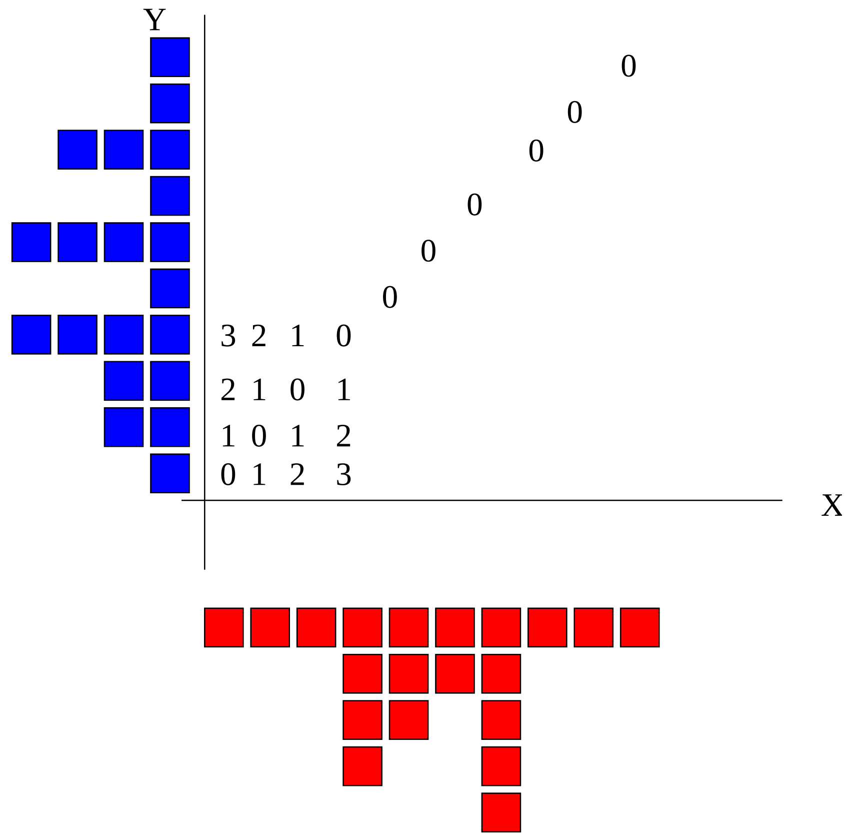
- γ positive, and marginals should be p and q

$$p(x) = \sum_{y \in Y} \gamma(x, y)$$
$$q(y) = \sum_{x \in X} \gamma(x, y)$$

Classical case

Optimal transport problem: a cost function

- $c : X \times Y \to \mathbb{R}^+$



- Cost: $\sum_{x \in X} \sum_{y \in Y} c(x,y)\gamma(x,y)$
- Distance: using the assignment with minimum cost.

Optimal transport and non-additive measures

Optimal transport

Optimal transport problem: The case of non-additive measures

- input:
  - X, and non-additive measure μ on X
  - Y, and non-additive measure ν on Y
- Output:
  - Assignment from μ to ν
  - A cost of the assignment: optimal

Optimal transport

How to proceed?

- Option 0. We consider a cost function on $X \times Y$ and a Choquet integral of measures on $X \times Y$ with marginals μ and ν.
  - For all non-additive measures in $X \times Y$, minimum CI

Optimal transport

How to proceed?

- Option 0. We consider a cost function on $X \times Y$ and a Choquet integral of measures on $X \times Y$ with marginals μ and ν.
  - For all non-additive measures in $X \times Y$, minimum CI
- The problem seems difficult in practice
  - The Fubini theorem does not apply in general for Choquet integral
  - Margins, also Choquet integrals (?)

Transforms

Measures and transforms: $\mu \leftrightarrow \tau_\mu$

- $(\max, +)$-transform[4]

$$\tau_\mu(B) = \mu(B) - \max_{A \subset B} \mu(A)$$

  - The $(\max, +)$-transform is always positive in [0,1]
  - If μ additive $\tau_\mu(B) = \min_{x_i \in B} \mu(\{x_i\})$.

[4] V. Torra, $(Max, \oplus)$-transforms and genetic algorithms for fuzzy measure identification, Fuzzy Sets and Systems 28 (2022) 253-265

Optimal transport

- Option 3. Definition through the $(max, +)$-transform, cost $c_a : 2^X \times 2^X \to [0, 1]$
  - Find $assg : 2^X \times 2^X \to [0, 1]$ such that
    - ▷ $assg(\emptyset, \emptyset) = 0$
    - ▷ $\tau_\mu(A) = \sum_{B' \subseteq X} assg(A, B')$ for all $A \neq \emptyset$
    - ▷ $\tau_\nu(B) = \sum_{A' \subseteq X} assg(A', B)$ for all $B \neq \emptyset$
  - Cost of the assignment:
    - ▷ $cost(c_a, assg) = \sum_{A \subseteq X} \sum_{B \subseteq X} c_a(A, B) assg(A, B)$.

Optimal transport

- Option 3. Then, we can define:
  - Optimal transport: Assignment with minimal cost
  - Wasserstein-like discrepancy:

$$d_{c_a}(\mu, \nu) = \inf_{assg \in \Pi(\tau_\mu, \tau_\nu)} \sum_{A \subseteq X} \sum_{B \subseteq X} c_a(A, B) assg(A, B)$$

Optimal transport

- Option 3. Example, μ is additive, ν is not, $(\max, +)$-transforms τ_μ and τ_ν, feasible assignment:

$\nu(B)$	τ_ν	set	$lack_\nu$							
1	0.8	X	0	0	0.3	0.5	0	0	0	0
0	0	$\{x_2, x_3\}$	0	0	0	0	0	0	0	0
0.2	0	$\{x_1, x_3\}$	0	0	0	0	0	0	0	0
0.2	0	$\{x_1, x_2\}$	0	0	0	0	0	0	0	0
0	0	$\{x_3\}$	0	0	0	0	0	0	0	0
0	0	$\{x_2\}$	0	0	0	0	0	0	0	0
0.2	0.2	$\{x_1\}$	0	0.2	0	0	0	0	0	0
$lack_\mu$		$\emptyset$	$--$	0	0	0	0.1	0.1	0.4	0.1
set			$\emptyset$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$	X
τ_μ	$--$		$--$	0.2	0.3	0.5	0.1	0.1	0.4	0.1
$\mu(A)$			$--$	0.2	0.3	0.5	0.4	0.6	0.9	1.0

Optimal transport

- Properties[5]:
  - This is a proper generalization
  - When our FM solution is a probability solution?
    - assignment on X, Y vs. assignment on 2^X, 2^Y
    - cost on X, Y vs. cost on 2^X, 2^Y
  - Results on
    - A cost function that is independent on the measures
    - A cost function that depends on the measures

[5] V. Torra (2023) The transport problem for non-additive measures, Euro. J Oper. Res. 311 679-689

Optimal transport

- Implementation: Linear problem with linear constraints
  - 1. Linear problem with linear constraints (case belief functions)
    OT with 2^X variables
  - 2. Linear problem with linear constraints (case Möbius transform)
    Same but: transformation of $|t|$ into two additional constraints $+t \leq t'$, $-t \leq t'$. So, $2 \cdot 2^{2|X|}$ additional constraints
  - 3. Linear problem with linear constraints (case $(\max, +)$-transform)
    OT with 2^X variables.
    Software: http://www.mdai.cat/code/
  - ▷ What is an appropriate cost function?

Summary

Summary

- Tools for measures and games
  - Υ-values and their characterization
  - ϕ and Υ-values for explainability
  - Optimal transport problem for non-additive measures

Thank you

References

- VT (2024) Υ-values: power indices à la orness for non-additive measures, IEEETFS.
- VT (2023) The transport problem for non-additive measures, Euro. J Oper. Res. 311 679-689.
- VT (2025) The complementary role of Shapley and Υ-values in explainability, submitted.
- VT (2026) Approximating fuzzy measures with distorted probabilities, forthcomming (probably).
- Narukawa, Y., VT (2005) Fuzzy measure and probability distributions: distorted probabilities, IEEE Trans. on Fuzzy Systems 13:5 617-629.
- VT (1997) The weighted OWA operator, IJIS 12 153-166.